On the Period of a Periodic-Finite-Type Shift

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***Abstract*— Periodic-finite-type shifts (PFT's) form a class of sofic shifts that strictly contains the class of shifts of finite type (SFT's). In this paper, we investigate how the notion of "period" inherent in the definition of a PFT causes it to differ from an SFT, and how the period influences the properties of a PFT.**

I. Introduction

Shifts of finite type (SFT's) are objects of fundamental importance in symbolic dynamics and the theory of constrained coding [2]. A well-known example of an SFT would be the (d,k) run-length limited ((d,k)-RLL) shift, where the number of 0's between successive 1's is at least d and at most k. Constrained codes based on these (d,k)-RLL shifts are used in most storage media such as magnetic tapes, CD's and DVD's.

A generalization of SFT's was introduced by Moision and Siegel [4] who were interested in examining the properties of distance-enhancing constrained codes, in which the appearance of certain words is forbidden in a periodic manner. This new class of shifts, called periodic-finite-type shifts (PFT's), contains the class of SFT's and some other interesting classes of shifts, such as constrained systems with unconstrained positions [1],[7], and shifts arising from the time-varying maximum transition run constraint [6]. The class of PFT's is in turn properly contained within the class of sofic shifts [3], a fact we discuss in more detail in Section II.

The properties of SFT's are now quite well understood (cf. [2]), but the same cannot be said for PFT's. The study of PFT's has primarily focused on finding efficient algorithms for constructing their presentations [1], [3], [5]. The difference between the definitions of SFT's and PFT's is quite small. An SFT is defined as a set of bi-infinite sequences (over some alphabet) that do not contain as subwords any word from a certain finite set. Thus, an SFT is defined by forbidding the appearance of finitely many words at any position of a bi-infinite sequence. A PFT is also defined by forbidding the appearance of finitely many words, except that these words are only forbidden to appear at positions of a bi-infinite sequence that are indexed by certain pre-defined periodic integer sequences; see Section II for a formal definition. This paper aims to initiate a study of how the "period" inherent in the definition of a PFT influences its properties.

After a review of relevant definitions and background in Section II, we will see in Section III that given an SFT $\mathcal{Y}$, we can associate with it a PFT $\mathcal{X}$ in such a way that it is only the period that differentiates $\mathcal{X}$ from $\mathcal{Y}$. We then seek to understand how the period determines the properties of the PFT $\mathcal{X}$ by means of a comparative study of $\mathcal{X}$ and $\mathcal{Y}$. We investigate a different aspect of periods in Section IV, where we study the influence of the period of a PFT $\mathcal{X}$ on the periods of periodic sequences in $\mathcal{X}$, and on the periods of graphical presentations of $\mathcal{X}$.

II. Basic Background on SFT's and PFT's

We begin with a review of basic background, based on material from [2] and [3]. Let Σ be a finite set of symbols; we call Σ an *alphabet*. We always assume that $|\Sigma| = q \geq 2$ since $q = 1$ gives us a trivial case. Let $\mathbf{w} = \ldots w_{-1}w_0w_1\ldots$ be a bi-infinite sequence over Σ. A word (finite-length sequence) $u \in \Sigma^n$ (for some integer n) is said to be a *subword* of $\mathbf{w}$, denoted by $u \prec \mathbf{w}$, if $u = w_i w_{i+1}\ldots w_{i+n-1}$ for some integer i. If we want to emphasize the fact that u is a subword of $\mathbf{w}$ starting at the index i, (*i.e.*, $u = w_i w_{i+1}\ldots w_{i+n-1}$), we write $u \prec_i \mathbf{w}$. By convention, we assume that the empty word $\epsilon \in \Sigma^0$ is a subword of any bi-infinite sequence. Also, we define σ to be the shift map, that is, $\sigma(\mathbf{w}) = \ldots w^*_{-1}w^*_0w^*_1\ldots$ is the bi-infinite sequence satisfying $w^*_i = w_{i+1}$ for all i.

Given a labeled directed graph $\mathcal{G}$, where labels come from Σ, let $S(\mathcal{G})$ be the set of bi-infinite sequences which are generated by reading off labels along bi-infinite paths in $\mathcal{G}$. A *sofic shift* $\mathcal{S}$ is a set of bi-infinite sequences such that $\mathcal{S} = S(\mathcal{G})$ for some labeled directed graph $\mathcal{G}$. In this case, we say that $\mathcal{S}$ is *presented by* $\mathcal{G}$, or that $\mathcal{G}$ is a *presentation* of $\mathcal{S}$. It is well known that every sofic shift has a *deterministic* presentation, *i.e.*, a presentation such that outgoing edges from the same state (vertex) are labeled distinctly. For a sofic shift $\mathcal{S}$, $\mathcal{B}_n(\mathcal{S})$ denotes the set of words $u \in \Sigma^n$ satisfying $u \prec \mathbf{w}$ for some bi-infinite sequence $\mathbf{w}$ in $\mathcal{S}$, and $\mathcal{B}(\mathcal{S}) = \cup_{n\geq 0}\mathcal{B}_n(\mathcal{S})$. A sofic shift $\mathcal{S}$ is *irreducible* if there is an irreducible (*i.e.*, strongly connected) presentation of $\mathcal{S}$, or equivalently, for every ordered pair of words u and v in $\mathcal{B}(\mathcal{S})$, there exists a word $z \in \mathcal{B}(\mathcal{S})$ such that $uzv \in \mathcal{B}(\mathcal{S})$.

A *shift of finite type* (SFT) $\mathcal{Y}_{\mathcal{F}'}$, with a finite set of forbidden words (a forbidden set) $\mathcal{F}'$, is the set of all bi-infinite sequences $\mathbf{w} = \cdots w_{-1}w_0w_1\cdots$ over Σ such that $\mathbf{w}$ contains no word $f' \in \mathcal{F}'$ as a subword. That is, the finite number of words f' in $\mathcal{F}'$ are not in $\mathcal{B}(\mathcal{Y}_{\mathcal{F}'})$. A *periodic-finite-type shift*, which we abbreviate as *PFT*, is characterized by an ordered list of finite sets $\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(T-1)})$

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and a *period* T. The PFT $\mathcal{X}_{\{\mathcal{F},T\}}$ is defined as the set of all bi-infinite sequences $\mathbf{w}$ over Σ such that for some integer $r \in \{0, 1, \ldots, T-1\}$, the r-shifted sequence $\sigma^r(\mathbf{w})$ of $\mathbf{w}$ satisfies $u \prec_i \sigma^r(\mathbf{w}) \implies u \notin \mathcal{F}^{(i \bmod T)}$ for every integer i. For simplicity, we say that a word f is in $\mathcal{F}$ (symbolically, $f \in \mathcal{F}$) if $f \in \mathcal{F}^{(j)}$ for some j. Since the appearance of words $f \in \mathcal{F}$ is forbidden in a periodic manner, note that f can be in $\mathcal{B}(\mathcal{X}_{\{\mathcal{F},T\}})$. Also, observe that a PFT $\mathcal{X}_{\{\mathcal{F},T\}}$ satisfying $\mathcal{F}^{(0)} = \mathcal{F}^{(1)} = \cdots = \mathcal{F}^{(T-1)}$ is simply the SFT $\mathcal{Y}_{\mathcal{F}'}$ with $\mathcal{F}' = \mathcal{F}^{(0)}$. Thus, SFT's are special cases of PFT's. We call a PFT *proper* when it cannot be represented as an SFT.

Any SFT can be considered to be an SFT in which every forbidden word has the same length. More precisely, given an SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}^*}$, find the longest forbidden word in $\mathcal{F}^*$ and say it has length ℓ. Set $\mathcal{F}' = \{f' \in \Sigma^\ell : f' \text{ has some } f^* \in \mathcal{F}^* \text{ as a prefix}\}$. Then, $\mathcal{Y}_{\mathcal{F}^*} = \mathcal{Y}_{\mathcal{F}'}$, and each word in $\mathcal{F}'$ has the same length, ℓ. Furthermore, we can also assume that $\mathcal{B}_\ell(\mathcal{Y}) = \Sigma^\ell \setminus \mathcal{F}'$ since if not (that is, if $\mathcal{B}_\ell(\mathcal{Y}) \subsetneq \Sigma^\ell \setminus \mathcal{F}'$), every word in $(\Sigma^\ell \setminus \mathcal{F}') \setminus \mathcal{B}_\ell(\mathcal{Y})$ can be added to $\mathcal{F}'$, without affecting $\mathcal{Y}$ in any way.

Correspondingly, every PFT $\mathcal{X}$ has a representation of the form $\mathcal{X}_{\{\mathcal{F},T\}}$ such that $\mathcal{F}^{(j)} = \emptyset$ for $1 \le j \le T-1$, and every word in $\mathcal{F}^{(0)}$ has the same length. An arbitrary representation $\mathcal{X}_{\{\mathcal{F},T\}}$ can be converted to one in the above form as follows. If $f \in \mathcal{F}^{(j)}$ for some $1 \le j \le T-1$, list out all words with length $j + |f|$ whose suffix is f, add them to $\mathcal{F}^{(0)}$, and delete f from $\mathcal{F}^{(j)}$. Continue this process until $\mathcal{F}^{(1)} = \cdots = \mathcal{F}^{(T-1)} = \emptyset$. Then, apply the method described above for SFT's to make every word in $\mathcal{F}^{(0)}$ have the same length.

It is known that PFT's belong to the class of sofic shifts.

Theorem II.1 (Moision and Siegel, [3]) *All periodic-finite-type shifts $\mathcal{X}$ are sofic shifts. That is, for any PFT $\mathcal{X}$, there is a presentation $\mathcal{G}$ of $\mathcal{X}$.*

Moision and Siegel proved the theorem by giving an algorithm that, given a PFT $\mathcal{X}$, generates a presentation, $\mathcal{G}_\mathcal{X}$, of $\mathcal{X}$. We call the presentation $\mathcal{G}_\mathcal{X}$ the *MS presentation* of $\mathcal{X}$. The *MS algorithm*, given a PFT $\mathcal{X}$ as input, runs as follows.

1) Represent $\mathcal{X}$ in the form $\mathcal{X}_{\{\mathcal{F},T\}}$, such that every word in $\mathcal{F}$ has the same length ℓ and belongs to $\mathcal{F}^{(0)}$.
2) Prepare T copies of Σ^ℓ and name them $\mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(T-1)}$.
3) Consider the words in $\mathcal{V}^{(0)}, \mathcal{V}^{(1)}, \ldots, \mathcal{V}^{(T-1)}$ as states. Draw an edge labeled $a \in \Sigma$ from $u = u_1 u_2 \cdots u_\ell \in \mathcal{V}^{(j)}$ to $v = v_1 v_2 \cdots v_\ell \in \mathcal{V}^{(j+1 \bmod T)}$ if and only if $u_2 \cdots u_\ell = v_1 \cdots v_{\ell-1}$ and $v_\ell = a$.
4) Remove states corresponding to words in $\mathcal{F}^{(0)}$ from $\mathcal{V}^{(0)}$, together with their incoming and outgoing edges. Call this labeled directed graph $\mathcal{G}'$.
5) If there is a state in $\mathcal{G}'$ having only incoming edges or only outgoing edges, remove the state from $\mathcal{G}'$ as well as its incoming or outgoing edges. Continue this process until we cannot find such a state. The resulting graph $\mathcal{G}_\mathcal{X}$ is a presentation of $\mathcal{X}$.

Remark II.2 *It is evident that the MS presentation of a PFT is always deterministic. Also, for a path α in $\mathcal{G}_\mathcal{X}$ with length $|\alpha| \ge \ell$, α terminates at some state that is a copy of $u = u_1 u_2 \ldots u_\ell$ iff the length-ℓ suffix of the word generated by α is equal to u.*

III. Influence of the Period T on a PFT

From this point on, whenever we consider an SFT $\mathcal{Y}_{\mathcal{F}'}$ in this paper, we will implicitly assume that every forbidden word in $\mathcal{F}'$ has the same length ℓ, and that $\mathcal{B}_\ell(\mathcal{Y}) = \Sigma^\ell \setminus \mathcal{F}'$. As we observed in the previous section, there is no loss of generality in doing so. Given an SFT $\mathcal{Y}_{\mathcal{F}'}$, consider the PFT $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ in which

$$\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(T-1)}) = (\mathcal{F}', \emptyset, \ldots, \emptyset).$$

While $\mathcal{Y}_{\mathcal{F}'} \subseteq \mathcal{X}_{\{\mathcal{F},T\}}$, equality does not hold in general. Note that it is only the influence of the period T that causes the shifts $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ and $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ to differ. So, a comparative study of $\mathcal{X}$ and $\mathcal{Y}$ is a useful means of understanding how the period T determines the properties of the PFT $\mathcal{X}$. In this section, we present a sampling of results that illustrate how properties of the SFT $\mathcal{Y}$ can affect those of the PFT $\mathcal{X}$.

The following result, which shows that the irreducibility of $\mathcal{Y}$ has a significant effect on the irreducibility of $\mathcal{X}$, may be considered typical of the comparative study proposed above.

Theorem III.1 *Suppose that $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ is an irreducible SFT. Let $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ be the PFT satisfying*

$$\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(T-1)}) = (\mathcal{F}', \emptyset, \ldots, \emptyset).$$

If there exists a periodic bi-infinite sequence $\mathbf{y}$ in $\mathcal{Y}$ with a period p satisfying $p \equiv 1 \pmod{T}$, then the MS presentation, $\mathcal{G}_\mathcal{X}$, of $\mathcal{X}$ is irreducible as a graph. That is, $\mathcal{X}$ is irreducible.

Proof: Throughout this proof, for a path η in a graph, let $s(\eta)$ and $t(\eta)$ be the starting state and the terminal state, respectively, of η in the graph. Also, for a state $v = v_1 v_2 \ldots v_\ell$ in $\mathcal{G}_\mathcal{X}$, $v \in \mathcal{V}^{(j)}$ is denoted by $v^{(j)}$ for $0 \le j \le T-1$.

Let $\mathcal{G}'$ be the graph defined in Step 4 of the MS algorithm. Consider the subgraph $\mathcal{H}$ of $\mathcal{G}'$ that is induced by the states in $\Sigma^\ell \setminus \mathcal{F}'$. Since $\Sigma^\ell \setminus \mathcal{F}' = \mathcal{B}_\ell(\mathcal{Y})$, all states in $\mathcal{H}$ have incoming edges and outgoing edges. Hence, $\mathcal{H}$ is a subgraph of $\mathcal{G}_\mathcal{X}$.

Key points of the proof are the following.

Claim 1: $\mathcal{H}$ is a presentation of $\mathcal{Y}$.

Claim 2: $\mathcal{H}$ is irreducible as a graph if there exists a periodic bi-infinite sequence $\mathbf{y}$ in $\mathcal{Y}$ with a period p satisfying $p \equiv 1 \pmod{T}$.

Once these claims are proved, it is straightforward to check that the MS presentation $\mathcal{G}_\mathcal{X}$ of $\mathcal{X}$ is irreducible. Note that the graph $\mathcal{G}'$ is obtained from $\mathcal{H}$ by adding words in $\mathcal{F}^{(0)}$ to $\mathcal{V}^{(1)}, \mathcal{V}^{(2)}, \ldots, \mathcal{V}^{(T-1)}$ and corresponding incoming and outgoing edges. Observe that (by Step 5 of the MS algorithm) a word $f' \in \mathcal{F}^{(0)}$ is a state in $\mathcal{G}_\mathcal{X}$ if and only if there exist paths ρ_1, ρ_2 in $\mathcal{G}'$ satisfying $s(\rho_1) = f'$, $t(\rho_1) \in \Sigma^\ell \setminus \mathcal{F}'$ and $s(\rho_2) \in \Sigma^\ell \setminus \mathcal{F}'$, $t(\rho_2) = f'$. Since $\mathcal{H}$ is irreducible, $\mathcal{G}_\mathcal{X}$ is irreducible as well.

Proof of Claim 1: We need to show that $S(\mathcal{H}) \subseteq \mathcal{Y}$ and

$\mathcal{Y} \subseteq S(\mathcal{H})$. It is clear that $S(\mathcal{H}) \subseteq \mathcal{Y}$ since, by Remark II.2, there is no path in $\mathcal{H}$ which generates words in $\mathcal{F}'$.

Conversely, take an arbitrary bi-infinite sequence $\mathbf{x} = \ldots x_{-1}x_0x_1 \ldots \in \mathcal{Y}$. Since $f' \not\prec \mathbf{x}$ for every forbidden word $f' \in \mathcal{F}'$, we see that for any integer i, the states corresponding to $x_{i-\ell+1}x_{i-\ell+2}\ldots x_i$ are in $\mathcal{H}$. Therefore, there exists an edge labeled x_{i+1} from $x_{i-\ell+1}x_{i-\ell+2}\ldots x_i \in \mathcal{V}^{(j)}$ to $x_{i-\ell+2}\ldots x_i x_{i+1} \in \mathcal{V}^{(j+1 \mod T)}$ for all integers i and $0 \le j \le T-1$. Hence, $\mathbf{x} \in S(\mathcal{H})$, that is, $\mathcal{Y} \subseteq S(\mathcal{H})$.

Proof of Claim 2: A periodic bi-infinite sequence $\mathbf{y} \in \mathcal{Y}$ with period $p \equiv 1 \pmod T$ can be written as $\mathbf{y} = (y_1y_2\ldots y_n)^\infty$, for some $y_1y_2\ldots y_n \in \Sigma^n$, where n is some multiple of p satisfying $n \equiv 1 \pmod T$ and $n \ge \ell$.

As $\mathbf{y} \in \mathcal{Y}$, $y_{n-\ell+1}\ldots y_n y_1 y_2 \ldots y_n \in \mathcal{B}(\mathcal{Y})$. Thus, for every $i \in \{0,1,\ldots,T-1\}$, there exists a path α in $\mathcal{H}$ satisfying $s(\alpha) = z^{(i)} = y_{n-\ell+1}\ldots y_n$ and generating $y_1y_2\ldots y_n$. Observe that $t(\alpha)$ is also $z^{(i')} = y_{n-\ell+1}\ldots y_n$ for some $i' \in \{0,1,\ldots,T-1\}$. However, since $|y_1y_2\ldots y_n| = n \equiv 1 \pmod T$, we have $i' = i+1 \mod T$. This automatically implies that for the word $z = y_{n-\ell+1}\ldots y_n$ in $\mathcal{B}(\mathcal{Y})$, there is a path β_{jk} in $\mathcal{H}$ such that $s(\beta_{jk}) = z^{(j)}$ and $t(\beta_{jk}) = z^{(k)}$ for any ordered pair (j,k), where $0 \le j,k \le T-1$.

Now take an arbitrary pair of states $u^{(r)}$ and $v^{(s)}$ in $\mathcal{H}$. Since $\mathcal{Y}$ is irreducible, there exist words w' and w^* in $\mathcal{B}(\mathcal{Y})$ so that $uw'z$ and zw^*v are in $\mathcal{B}(\mathcal{Y})$. Thus, there exists a path γ generating $w'z$ such that $s(\gamma) = u^{(r)}$ and $t(\gamma) = z^{(j)}$ for some $0 \le j \le T-1$, and a path δ generating w^*v such that $s(\delta) = z^{(k)}$ for some $0 \le k \le T-1$ and $t(\delta) = v^{(s)}$. As there is a path β_{jk} from $z^{(j)}$ to $z^{(k)}$ from the argument above, we have a path $\gamma\beta_{jk}\delta$ starting from $u^{(r)}$ and terminating at $v^{(s)}$. Hence, the presentation $\mathcal{H}$ is irreducible as a graph. ■

From Theorem III.1, we can obtain the following corollary.

Corollary III.2 *Let $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ be an irreducible SFT such that $|\mathcal{F}'| < |\Sigma|$. Then for all $T \ge 1$, the PFT $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ with*

$$\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(T-1)}) = (\mathcal{F}', \emptyset, \ldots, \emptyset)$$

is irreducible.

Proof: Since $|\mathcal{F}'| < |\Sigma|$, there is a symbol $a \in \Sigma$ which is not used as the first symbol of any word in $\mathcal{F}'$. Hence, the bi-infinite sequence $\mathbf{a} = a^\infty$ is in $\mathcal{Y}$. As $\mathbf{a}$ has period 1, we have from Theorem III.1 that $\mathcal{X}$ is irreducible. ■

The proof of Theorem III.1 shows that the SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ has a presentation $\mathcal{H}$ that is a subgraph of the MS presentation $\mathcal{G}_\mathcal{X}$ of $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$, where $\mathcal{F} = (\mathcal{F}', \emptyset, \ldots, \emptyset)$. This fact may allow us to compare some of the invariants associated with the two shifts $\mathcal{Y}$ and $\mathcal{X}$, for example, their entropies and their zeta functions (see [2, Chapters 4 and 6]). The entropy (or the Shannon capacity) $h(\mathcal{S})$ of a sofic shift $\mathcal{S}$ can be computed from a deterministic presentation $\mathcal{G}$ of $\mathcal{S}$ as follows: $h(\mathcal{S}) = \log_2 \lambda$, where λ is the largest eigenvalue of the adjacency matrix $A_\mathcal{G}$ of $\mathcal{G}$. Equivalently, λ is the largest root of the characteristic polynomial $\chi_{A_\mathcal{G}}(t) = \det(tI - A_\mathcal{G})$ of $A_\mathcal{G}$ (see, *e.g.*, [2, Chapter 4]).

Returning to the shifts $\mathcal{X}$ and $\mathcal{Y}$ as above, since $\mathcal{H}$ is a subgraph of $\mathcal{G}_\mathcal{X}$, it may be possible to express the characteristic polynomial of $A_{\mathcal{G}_\mathcal{X}}$ in terms of the characteristic polynomial of $A_\mathcal{H}$. This would allow us to compare the entropies of $\mathcal{X}$ and $\mathcal{Y}$. However, this seems to be hard to do in general. We have a partial result in the special case when $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ with $|\mathcal{F}'| = 1$, and $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},2\}}$, as we describe next.

Recall that $|\Sigma| = q$. Now suppose that $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ is an SFT with the set $\mathcal{F}'$ consisting of a single forbidden word f', and $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},2\}}$ is the PFT with period 2 and $\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}) = (\{f'\}, \emptyset)$. Also, let $A_{\mathcal{G}_\mathcal{X}}$ be the adjacency matrix of the MS presentation $\mathcal{G}_\mathcal{X}$ of $\mathcal{X}$, and let $A_\mathcal{H}$ be that of the subgraph $\mathcal{H}$ of $\mathcal{G}_\mathcal{X}$ induced by the states in $\Sigma^\ell \setminus \{f'\}$. Observe that the matrix $A_{\mathcal{G}_\mathcal{X}}$ is a $(2q^\ell - 1) \times (2q^\ell - 1)$ 0-1 matrix. Without loss of generality, for $A_{\mathcal{G}_\mathcal{X}}$, we can assume the following.

- The first $q^\ell - 1$ rows and columns correspond to states in $\mathcal{V}^{(0)}$, and the last q^ℓ rows and columns correspond to those in $\mathcal{V}^{(1)}$.
- Assign $f' \in \mathcal{V}^{(1)}$ to the $(2q^\ell - 1)$-th row and column, and arrange the first row so that the $(1, 2q^\ell - 1)$-th entry of $A_{\mathcal{G}_\mathcal{X}}$ is 1.
- Let $u \in \mathcal{V}^{(1)}$ be such that the longest proper suffix of u is equal to that of f'. Assign this u to the q^ℓ-th row and column so that the q^ℓ-th row and the $(2q^\ell - 1)$-th row are the same.

For a matrix M, set $M^{(i,j)}$ to be the submatrix of M obtained by deleting its i-th row and j-th column. Then, observe that $A_{\mathcal{G}_\mathcal{X}}^{(2q^\ell-1, 2q^\ell-1)} = A_\mathcal{H}$. In this case, by applying elementary row operations to the matrix $N = tI - A_{\mathcal{G}_\mathcal{X}}$, we have

$$\chi_{A_{\mathcal{G}_\mathcal{X}}}(t) = \det(N) = \begin{vmatrix} B & \mathbf{c} \\ \mathbf{d} & t \end{vmatrix}, \tag{1}$$

where B is a $(2q^\ell - 2) \times (2q^\ell - 2)$ matrix satisfying $\det(B) = \chi_{A_\mathcal{H}}(t)$, $\mathbf{c}$ is the $(2q^\ell - 2) \times 1$ column vector $[-1\ 0\ \ldots\ 0]^T$, and $\mathbf{d} \in \{-1, 0\}^{2q^\ell - 2}$. Using the form given in (1) for $\det(N)$, we can derive the following theorem. The complete proof will be published in the full version of this paper.

Theorem III.3 *Let $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ and $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},2\}}$ be the SFT and PFT described above, respectively. Then, the characteristic polynomial $\chi_{A_{\mathcal{G}_\mathcal{X}}}(t)$ of the adjacency matrix $A_{\mathcal{G}_\mathcal{X}}$ is given by*

$$\chi_{A_{\mathcal{G}_\mathcal{X}}}(t) = t(\chi_{A_\mathcal{H}}(t) + (-1)^{q^\ell} \det(B^{(1,q^\ell)})).$$

IV. Periods in PFT's

The period T involved in the description of a PFT is not the only notion of "period" that can be associated with the shift. For any shift $\mathcal{X}$, we can always define its *sequential period*, $T_{seq}^{(\mathcal{X})}$, to be the smallest period of any periodic bi-infinite sequence in $\mathcal{X}$. Furthermore, if $\mathcal{X}$ is an irreducible sofic shift, we can define a "graphical period" for it as follows. Let $\mathcal{G}$ be a presentation of $\mathcal{X}$ with state set $\mathcal{V}(\mathcal{G}) = \{V_1, \ldots, V_r\}$. For each $V_i \in \mathcal{V}(\mathcal{G})$, define $\text{per}(V_i)$ to be the greatest common divisor (gcd) of the lengths of paths (cycles) in $\mathcal{G}$ that begin and end at V_i, and further define

$\text{per}(\mathcal{G}) = \gcd(\text{per}(V_1), \ldots, \text{per}(V_r))$. It is well known that when $\mathcal{G}$ is irreducible, $\text{per}(V_i) = \text{per}(V_j)$ for each pair of states $V_i, V_j \in \mathcal{V}(\mathcal{G})$, and hence $\text{per}(\mathcal{G}) = \text{per}(V)$ for any $V \in \mathcal{V}(\mathcal{G})$. The *graphical period*, $T_{graph}^{(\mathcal{X})}$, of an irreducible sofic shift $\mathcal{X}$ is defined to be the least $\text{per}(\mathcal{G})$ of any irreducible presentation $\mathcal{G}$ of $\mathcal{X}$.

Given a PFT $\mathcal{X}$, define its *descriptive period*, $T_{desc}^{(\mathcal{X})}$, to be the smallest integer among all T^* such that $\mathcal{X} = \mathcal{X}_{\{\mathcal{F}^*, T^*\}}$ for some $\mathcal{F}^*$. In this section, we determine what influence, if any, the descriptive period of a PFT has on its sequential and graphical periods.

Let $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ be an irreducible PFT, and let $\mathcal{G}$ be an irreducible presentation of $\mathcal{X}$. Proposition 1 of [3] says that if $\mathcal{X}$ is proper, then $\gcd(\text{per}(\mathcal{G}), T) \neq 1$. Using that proposition, we can obtain the following result, which shows that a proper PFT $\mathcal{X}$ can have $T_{desc}^{(\mathcal{X})}$ arbitrarily larger than $T_{seq}^{(\mathcal{X})}$.

Proposition IV.1 *Suppose that $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ is an irreducible SFT, such that the bi-infinite sequence $a^\infty \in \mathcal{Y}$ for some $a \in \Sigma$. Let $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ be the PFT satisfying*

$$\mathcal{F} = (\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(T-1)}) = (\mathcal{F}', \emptyset, \ldots, \emptyset).$$

Then, $a^\infty \in \mathcal{X}$, so $T_{seq}^{(\mathcal{X})} = 1$. Furthermore, if $\mathcal{X}$ is a proper PFT and T is prime, we have $T_{desc}^{(\mathcal{X})} = T$.

Proof: Since $\mathcal{Y} \subseteq \mathcal{X}$, it is clear that $a^\infty \in \mathcal{X}$, and hence, $T_{seq}^{(\mathcal{X})} = 1$. Now, let $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ be a proper PFT with T prime. First observe that the MS presentation $\mathcal{G}_\mathcal{X}$ of $\mathcal{X}$ is irreducible since the bi-infinite sequence $\mathbf{a} = a^\infty$ is in $\mathcal{Y}$ and $\mathbf{a}$ has period 1. Also, note that $\text{per}(\mathcal{G}_\mathcal{X})$ must be kT for some $k \geq 1$ from the construction of $\mathcal{G}_\mathcal{X}$. However, if we consider the period of the states a^ℓ in $\mathcal{G}_\mathcal{X}$, it is T. Thus, $\text{per}(\mathcal{G}_\mathcal{X}) = T$ by the irreducibility of $\mathcal{G}_\mathcal{X}$. Since $\mathcal{X}$ is proper, we have from Proposition 1 of [3] that $\gcd(\text{per}(\mathcal{G}_\mathcal{X}), T^*) \neq 1$ for all T^* satisfying $\mathcal{X} = \mathcal{X}_{\{\mathcal{F}^*, T^*\}}$. As T is prime, $\gcd(\text{per}(\mathcal{G}_\mathcal{X}), T') = \gcd(T, T') = 1$ for all $T' < T$. Therefore, T is the descriptive period of $\mathcal{X}$. ■

For example, consider an SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$ with a forbidden set $\mathcal{F}' = \{b^2\}$ for some $b \in \Sigma$. Then, $\mathcal{Y}$ is irreducible, and $a^\infty \in \mathcal{Y}$ for any $a \in \Sigma \setminus \{b\}$. In this case, for a PFT $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},T\}}$ with T prime, such that $\mathcal{F} = (\{b^2\}, \emptyset, \ldots, \emptyset)$, it may be verified that $\mathcal{X}$ is proper, and hence, $T = T_{desc}^{(\mathcal{X})}$.

Conversely, $T_{seq}^{(\mathcal{X})}$ can be arbitrarily larger than $T_{desc}^{(\mathcal{X})}$ for proper PFT's $\mathcal{X}$. We present such an example next.

Set $\Sigma = \{0, 1\}$. We define a sliding-block map ψ as follows: for a non-empty word $u = u_1 u_2 \ldots u_n \in \Sigma^n$, (resp. a bi-infinite sequence $\mathbf{w} = \ldots w_{-1} w_0 w_1 \ldots$ over Σ), define $\psi(u) = u_1^* u_2^* \ldots u_{n-1}^*$, where $u_i^* = u_i + u_{i+1} \pmod 2$ for $1 \leq i \leq n-1$ (resp. $\psi(\mathbf{w}) = \ldots w_{-1}^* w_0^* w_1^* \ldots$, where $w_i^* = w_i + w_{i+1} \pmod 2$ for each i). By convention, $\psi(u) = \epsilon$ when $u \in \Sigma^1$. For $k \geq 1$, consider the PFT $\mathcal{X}_k = \mathcal{X}_{\{\mathcal{F}_k, 2\}}$ with $\mathcal{F}_k = (\mathcal{F}_k^{(0)}, \mathcal{F}_k^{(1)})$, defined as follows.

- $\mathcal{F}_k^{(1)} = \emptyset$ for all $k \geq 1$.
- $\mathcal{F}_1^{(0)} = \{0\}$, and for $k \geq 2$, we set $\mathcal{F}_k^{(0)} = \psi^{-1}(\mathcal{F}_{k-1}^{(0)})$. That is, $\mathcal{F}_k^{(0)}$ is the inverse image of $\mathcal{F}_{k-1}^{(0)}$ under ψ.

It is easy to see that for each $k \geq 1$, every word $f \in \mathcal{F}_k^{(0)}$ has length $|f| = k$, and in particular, we have $0^k \in \mathcal{F}_k^{(0)}$. Moreover, as ψ is a two-to-one mapping, we have $|\mathcal{F}_k^{(0)}| = 2^{k-1}$. The following proposition contains another useful observation concerning ψ. We omit the straightforward proof by induction.

Proposition IV.2 *For a binary word $u = u_1 u_2 \ldots u_r$ of length $r > m$, let $u_1^* u_2^* \ldots u_{r-m}^* = \psi^m(u)$. If $m = 2^j$ for some $j \geq 0$, then $u_i^* = u_i + u_{i+2^j} \pmod 2$ for $1 \leq i \leq r-m$. Furthermore, if $m = 2^j - 1$ for some $j \geq 0$, then $u_i^* = u_i + u_{i+1} + \cdots + u_{i+2^j-1} \pmod 2$ for $1 \leq i \leq r-m$.*

The corollary below simply follows from the fact that for any $f \in \mathcal{F}_k^{(0)}$, we must have $\psi^{k-1}(f) = 0$.

Corollary IV.3 *If $z \in \Sigma^{2^j}$ (for some $j \geq 0$) has an odd number of 1's, then $z \notin \mathcal{F}_{2^j}^{(0)}$.*

We next record some important facts about the PFT's $\mathcal{X}_k$.

Proposition IV.4 *For $k \geq 1$, the following statements hold: (a) $\mathcal{X}_{k+1} = \psi^{-1}(\mathcal{X}_k)$; (b) $\mathcal{X}_k$ is irreducible iff $1 \leq k \leq 6$; and (c) $\mathcal{X}_k$ is a proper PFT.*

Proof: Statement (a) follows straightforwardly from the definition of the PFT's $\mathcal{X}_k$.

For (b), first note that $\mathcal{X}_k$ is irreducible for $1 \leq k \leq 6$ since its MS presentation may be verified to be irreducible as a graph. When $k = 7$, it can be shown that $\mathcal{X}_k$ is not irreducible, which implies that $\mathcal{X}_k$ is not irreducible when $k \geq 7$ by (a).

To prove (c), suppose to the contrary that $\mathcal{X}_k$ is not a proper PFT for some $k \geq 1$. Then, $\mathcal{X}_k = \mathcal{Y}$ for some SFT $\mathcal{Y} = \mathcal{Y}_{\mathcal{F}'}$, where every forbidden word in $\mathcal{F}'$ has the same length, ℓ. Pick a $j \geq 0$ such that $2^j \geq k$, and set $r = 2^j - k$. By (a) above, $\mathcal{X}_{2^j} = \psi^{-r}(\mathcal{X}_k) = \psi^{-r}(\mathcal{Y})$. Note that $\psi^{-r}(\mathcal{Y})$ is also an SFT, with forbidden set $\psi^{-r}(\mathcal{F}')$. All words in $\psi^{-r}(\mathcal{F}')$ have length $\ell' = \ell + r$.

For the PFT $\mathcal{X}_{2^j}$, observe that the bi-infinite sequence $\mathbf{w} = (0^{2^j-1}1)^\infty 0^{2^j} (10^{2^j-1})^\infty$ is in $\mathcal{X}_{2^j}$ as $\mathbf{w}$ contains a word in $\mathcal{F}_{2^j}^{(0)}$ (*i.e.*, 0^{2^j}) only once, by Corollary IV.3. Therefore, every subword of $\mathbf{w}$ is in $\mathcal{B}(\mathcal{X}_{2^j}) = \mathcal{B}(\psi^{-r}(\mathcal{Y}))$.

Now, consider the bi-infinite sequence

$$\mathbf{w}' = (0^{2^j-1}1)^\infty 0^{2^j} (10^{2^j-1})^{2\ell'+1} 10^{2^j} (10^{2^j-1})^\infty.$$

Note that every length-ℓ' subword of $\mathbf{w}'$ is also a subword of $\mathbf{w}$, and hence, is in $\mathcal{B}(\psi^{-r}(\mathcal{Y}))$. This implies that $\mathbf{w}' \in \psi^{-r}(\mathcal{Y})$. For the two distinct indices m, n $(m < n)$ such that $0^{2^j} \prec_m \mathbf{w}'$ and $0^{2^j} \prec_n \mathbf{w}'$, we have $n - m = 2^j(2\ell' + 2) + 1$, so that $m \not\equiv n \pmod 2$. But, since $0^{2^j} \in \mathcal{F}_{2^j}^{(0)}$, this implies that $\mathbf{w}' \notin \mathcal{X}_{2^j}$, which is a contradiction. ■

Statement (c) of Proposition IV.4 implies that $T_{desc}^{(\mathcal{X}_k)} = 2$ for all $k \geq 1$. In contrast, the following theorem shows that $T_{seq}^{(\mathcal{X}_k)}$ grows arbitrarily large as $k \to \infty$.

Theorem IV.5 *For any $j \geq 0$ and $2^j + 1 \leq k \leq 2^{j+1}$, the periods of periodic sequences in $\mathcal{X}_k$ must be multiples of 2^{j+1}.*

To prove Theorem IV.5, we need the next three lemmas. We omit the simple proof of the first lemma.

Lemma IV.6 *If $\mathbf{x} \in \{0,1\}^{\mathbb{Z}}$ is a periodic sequence, then so is $\psi(\mathbf{x})$. Furthermore, any period of $\mathbf{x}$ is also a period of $\psi(\mathbf{x})$.*

Lemma IV.7 *For any $j \geq 0$, $\mathcal{F}^{(0)}_{2^j+1} = \{f^* f^*_1 : f^* = f^*_1 f^*_2 \ldots f^*_{2^j} \in \Sigma^{2^j}\}$.*

Proof: Recall that for a word $f \in \mathcal{F}^{(0)}_{2^j+1}$, $\psi^{2^j}(f) = 0$. Since Proposition IV.2 shows that $\psi^{2^j}(f) = f_1 + f_{2^j+1} \pmod 2$, we have $f_1 = f_{2^j+1}$. Noting that $|\mathcal{F}^{(0)}_{2^j+1}| = 2^{2^j} = |\Sigma^{2^j}|$, we thus have $\mathcal{F}^{(0)}_{2^j+1} = \{f^* f^*_1 : f^* = f^*_1 f^*_2 \ldots f^*_{2^j} \in \Sigma^{2^j}\}$. ∎

Lemma IV.8 *For $j \geq 0$, there is no periodic sequence $\mathbf{x}$ in $\mathcal{X}_{2^j+1}$ whose period is $(2t+1)2^j$ for some $t \geq 0$.*

Proof: We deal with $j = 0$ first. Note that $\mathcal{F}^{(0)}_2 = \{00, 11\}$. So, if $\mathcal{X}_2$ has a periodic bi-infinite sequence $\mathbf{w} = (w_1 w_2 \ldots w_m)^\infty$ with an odd period m, then $00 \not\prec w_1 w_2 \ldots w_m$, $11 \not\prec w_1 w_2 \ldots w_m$, and $w_1 \neq w_m$. But there is no word $w_1 w_2 \ldots w_m \in \Sigma^m$ that satisfies these conditions.

Now, consider $j \geq 1$. Assume, to the contrary, that there exists a periodic sequence $\mathbf{x} = \ldots x_{-1} x_0 x_1 \ldots \in \mathcal{X}_{2^j+1}$ whose period is $(2t+1)2^j$ for some $t \geq 0$. Then, $\mathbf{x}$ is of the form $(x_0 x_1 \ldots x_{(2t+1)2^j-1})^\infty$. Without loss of generality, we may assume that for every even integer i, $u \prec_i \mathbf{x}$ implies $u \notin \mathcal{F}^{(0)}_{2^j+1}$. Then, for each integer m, $x_{m2^j} x_{m2^j+1} \ldots x_{(m+1)2^j} \notin \mathcal{F}^{(0)}_{2^j+1}$. So, by Lemma IV.7, we have $x_{m2^j} \neq x_{(m+1)2^j}$. This implies that $x_0 = x_{(2t)2^j}$ as $|\Sigma| = 2$. But then, $x_{(2t)2^j} \ldots x_{(2t+1)2^j-1} x_0 \in \mathcal{F}^{(0)}_{2^j+1}$, which is a contradiction. ∎

We are now in a position to prove Theorem IV.5.

Proof of Theorem IV.5: To prove the theorem, it is enough to show that for $j \geq 0$, the periods of periodic sequences in $\mathcal{X}_{2^j+1}$ must be multiples of 2^{j+1}. It then follows, by Lemma IV.6, that the same also applies to periodic sequences in $\mathcal{X}_k$, for $2^j + 1 < k \leq 2^{j+1}$.

When $j = 0$, the required statement clearly holds by Lemma IV.8. So, suppose that the statement is true for some $j \geq 0$, so that periodic sequences in $\mathcal{X}_{2^j+1}$ have only multiples of 2^{j+1} as periods. Therefore, by Lemma IV.6, periodic sequences in $\mathcal{X}_{2^{j+1}+1}$ also can only have multiples of 2^{j+1} as periods. However, by Lemma IV.8, no periodic sequence in $\mathcal{X}_{2^{j+1}+1}$ can have an odd multiple of 2^{j+1} as a period. Hence, all periodic sequences in $\mathcal{X}_{2^{j+1}+1}$ have periods that are multiples of 2^{j+2}. The theorem follows by induction. ∎

Theorem IV.5 shows that for $2^j + 1 \leq k \leq 2^{j+1}$, we have $T^{(\mathcal{X}_k)}_{seq} \geq 2^{j+1}$. In fact, this holds with equality.

Corollary IV.9 *$T^{(\mathcal{X}_1)}_{seq} = 1$, and for $k \geq 2$, if $j \geq 0$ is such that $2^j + 1 \leq k \leq 2^{j+1}$, then $T^{(\mathcal{X}_k)}_{seq} = 2^{j+1}$.*

Proof: When $k = 1$, $T^{(\mathcal{X}_1)}_{seq} = 1$ as $1^\infty \in \mathcal{X}_1$. So let $k \geq 2$, and let $j \geq 0$ be such that $2^j + 1 \leq k \leq 2^{j+1}$. We only need to show that $T^{(\mathcal{X}_k)}_{seq} \leq 2^{j+1}$. The bi-infinite sequence $\mathbf{w} = (0^{2^{j+1}-1}1)^\infty$ is in $\mathcal{X}_{2^{j+1}}$ since, by Corollary IV.3, $\mathbf{w}$ contains no word in $\mathcal{F}^{(0)}_{2^{j+1}}$ as a subword. Since $\mathbf{w}$ has period 2^{j+1}, by Lemma IV.6, $\mathbf{w}' = \psi^{2^{j+1}-k}(\mathbf{w}) \in \mathcal{X}_k$ has period 2^{j+1} as well. Thus, $T^{(\mathcal{X}_k)}_{seq} \leq 2^{j+1}$. ∎

Theorem IV.5 also implies the following corollary.

Corollary IV.10 *$T^{(\mathcal{X}_k)}_{graph} \geq T^{(\mathcal{X}_k)}_{seq}$ holds when $1 \leq k \leq 6$.*

Proof: Since $\mathcal{X}_1$ is proper, $T^{(\mathcal{X}_1)}_{graph} \geq 2$ by Proposition 1 in [3]. Thus, $T^{(\mathcal{X}_1)}_{graph} > T^{(\mathcal{X}_1)}_{seq} = 1$.

So, let $k \geq 2$ and suppose $2^j + 1 \leq k \leq 2^{j+1}$ for some $j \geq 0$. By Corollary IV.9, we have $T^{(\mathcal{X}_k)}_{seq} = 2^{j+1}$. On the other hand, for any irreducible presentation $\mathcal{G}$ of $\mathcal{X}_k$, we have per$(\mathcal{G}) \geq 2^{j+1}$. Indeed, for each vertex V in $\mathcal{G}$, we have per(V) being a multiple of 2^{j+1}; otherwise we would have a contradiction of Theorem IV.5. Hence, $T^{(\mathcal{X}_k)}_{graph} \geq 2^{j+1} = T^{(\mathcal{X}_k)}_{seq}$ as required. ∎

Corollary IV.9 shows that $T^{(\mathcal{X}_k)}_{seq}$ grows arbitrarily large as $k \to \infty$, while $T^{(\mathcal{X}_k)}_{desc} = 2$ for all k. It also follows from Corollary IV.10 that $T^{(\mathcal{X}_k)}_{graph}$ is strictly larger than $T^{(\mathcal{X}_k)}_{desc}$ when $3 \leq k \leq 6$. Equality can hold in Corollary IV.10 — for example, when $k = 2$. Indeed, $\mathcal{X}_2$ is proper, and its MS presentation, $\mathcal{G}_{\mathcal{X}_2}$, is irreducible, with per$(\mathcal{G}_{\mathcal{X}_2}) = 2$, so that $T^{(\mathcal{X}_2)}_{graph} = 2$. From Corollary IV.9, we also have $T^{(\mathcal{X}_2)}_{seq} = 2$. Thus, $\mathcal{X}_2$ is an example of a proper PFT $\mathcal{X}$ in which $T^{(\mathcal{X})}_{seq} = T^{(\mathcal{X})}_{graph} = T^{(\mathcal{X})}_{desc}$ holds.

Thus, to summarize, there appears to be no relationship between the descriptive period of a PFT and its sequential period, as we have examples where each of these can be arbitrarily larger than the other. We have also found that, for a PFT $\mathcal{X}$, $T^{(\mathcal{X})}_{graph}$ can be larger than $T^{(\mathcal{X})}_{desc}$. However, we believe that the reverse cannot hold; in fact, we conjecture that $T^{(\mathcal{X})}_{desc}$ divides $T^{(\mathcal{X})}_{graph}$ for any PFT $\mathcal{X}$.

Finally, we note that we also have examples of proper PFT's $\mathcal{X}$ where $T^{(\mathcal{X})}_{seq}$ is arbitrarily larger than $T^{(\mathcal{X})}_{graph}$. We omit the proof due to space constraints.

Theorem IV.11 *Set $\Sigma = \{0,1\}$ and $k \geq 2$, and let $\mathcal{P}$ denote the set of all periodic bi-infinite sequences over Σ with period $k!$. Consider the PFT $\mathcal{X} = \mathcal{X}_{\{\mathcal{F},2\}}$ with $\mathcal{F} = (\mathcal{F}^{(0)}, \emptyset)$, such that $\mathcal{F}^{(0)} = \{w \in \Sigma^{2k!} : \exists \mathbf{x} \in \mathcal{P}$ such that $w \prec \mathbf{x}\}$. The following statements hold: (a) $\mathcal{X}$ is proper; (b) $\mathcal{X}$ is irreducible; and (c) $T^{(\mathcal{X})}_{seq} \geq k+1$ and $T^{(\mathcal{X})}_{graph} = 2$.*

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